Brief Review of Trigonometry

1. (a) An angle is determined by rotating a ray about its endpoint. (counterclockwise rotation measures a positive angle, clockwise rotation measures a negative angle.) Angles are measured by either degrees or radians. In Calculus, radian measure is used. One degree (1°) is the measure of a positive angle formed by 1/360 of one complete revolution and one radian is the measure of a positive angle which intercepts an arc $S$ which is equal in length to the radius $r$ of the circle. Hence, a central angle of $\theta$ radians intercepts an arc of length $\theta$ times the radius.

Formula for arc length

$$S = r \theta$$

(b) The area of a circular sector is a fractional part ($\theta/2\pi$) of the area of the circle ($\pi r^2$)

Hence

$$A = \left( \frac{\theta}{2\pi} \right) \pi r^2 = \frac{r^2 \theta}{2}$$

for the area of a circular sector.

(c) An angle of $1^\circ$ subtends an arc which is 1/360 of the circumference ($2\pi$) of the unit circle so that

$$1^\circ = \left( \frac{1}{360} \right) \pi \text{ radians} = \left( \frac{\pi}{180} \right) \text{ radians}$$

Example $30^\circ = 30 \left( \frac{\pi}{180} \text{ radians} \right) = \frac{\pi}{6} \text{ radians}$.

2. You must be able to find in exact form (no calculators) the trig functions for the special and quadrantal angles measured in radians. Below are some guidelines for remembering these.

(a) The circular functions are defined on a unit circle as follows:

$$\sin \theta = \frac{y}{r}, \quad \cos \theta = \frac{x}{r}, \quad \tan \theta = \frac{y}{x}, \quad \csc \theta = \frac{1}{y}, \quad \sec \theta = \frac{1}{x}, \quad \cot \theta = \frac{1}{\tan \theta}$$

By similar triangles we have the following definitions on a circle of radius $r$

and can relate this to the right triangle involved:

$$\sin \theta = \frac{y}{r}, \quad \cos \theta = \frac{x}{r}, \quad \tan \theta = \frac{y}{x}, \quad \csc \theta = \frac{r}{y}, \quad \sec \theta = \frac{r}{x}, \quad \cot \theta = \frac{x}{y}$$

(b) The triangles below from geometry can be used to remember the values of the trig functions at $30^\circ$ (π/6 radians), $45^\circ$ (π/4 radians), $60^\circ$ (π/3 radians)

Example:

$$\tan 45^\circ = \tan \left( \frac{\pi}{4} \text{ radians} \right) = \frac{\text{opp}}{\text{adj}} = 1 / 1 = 1$$

$$\cos 60^\circ = \cos \left( \frac{\pi}{3} \text{ radians} \right) = \frac{\text{adj}}{\text{hyp}} = 1 / 2$$

(c) For angles which are multiples of $\pi/6$, $\pi/4$, $\pi/3$ we use the concept of reference angles. The reference angle (called $\alpha$ here) is the acute angle formed by the terminal side of the given angle and the x-axis. The value for a circular function of the given angle $\theta$ is the same in absolute value as the value of the circular function of the reference angle. The sign of the circular function of the given angle is determined by the particular quadrant in which the terminal side of the given angle lies. Since $\alpha$ will be an acute angle, we can use the triangles above.

**Quadrant II** Here $\alpha$ is the reference angle and $\alpha = \pi - \theta$

$$\sin \alpha = \sin (\pi - \theta) = \sin \theta$$

$$\cos \alpha = -\cos \theta$$

$$\tan \alpha = -\tan \theta$$

**Quadrant III** Here $\alpha$ is the reference angle $\alpha = \theta - \pi$

$$\sin \alpha = -\sin \theta$$

$$\cos \alpha = -\cos \theta$$

$$\tan \alpha = \tan \theta$$

**Quadrant IV** Here $\alpha$ is the reference angle $\alpha = 2\pi - \theta$

$$\sin \alpha = \sin (2\pi - \theta) = -\sin \theta$$

$$\cos \alpha = \cos \theta$$

$$\tan \alpha = -\tan \theta$$
3. For the quadrant angles: $0, \pi/2, \pi, 3\pi/2, 2\pi$, etc. the graphs are useful.

4. The trigonometric functions evaluated at the real number $t$ are defined to have the same values as the corresponding circular functions for an angle of $t$ radians.
Ex: $\sin 2 \pi = \sin(2\text{ rad}) = + \sin(\pi - 2)\text{ rad} = + \sin(\pi - 2)$. Here $t$ is 2 and the reference angle $\alpha$ is $\pi - 2$ since $\pi/2 \leq 2 \leq \pi$

5. (a) An even function has the property that $f(-x) = f(x)$. The cosine function is even so $\cos(-x) = \cos(x)$.
Ex. $\cos(-\pi/6) = \cos(\pi/6) = \sqrt{3}/2$

(b) An odd function has the property that $f(-x) = -f(x)$. The sine function is odd so $\sin(-x) = -\sin(x)$
Ex. $\sin(-\pi/3) = -\sin(\pi/3) = -\sqrt{3}/2$

6. With the guidelines above you should be able to evaluate the trig functions for the chart below without a calculator.

![Chart](image)

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7. Trig identities that you must know:

Pythagorean identities
- $\sin^2 \theta + \cos^2 \theta = 1$
- $\tan^2 \theta + 1 = \sec^2 \theta$
- $\cot^2 \theta + 1 = \csc^2 \theta$

Double Angle identities
- $\sin 2\theta = 2 \sin \theta \cos \theta$
- $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$
- $\cot 2\theta = \frac{1 + \cos 2\theta}{\sin 2\theta}$
- $\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$
- $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$

8. Identities used less often

Sums and differences
- $\sin(\theta_1 \pm \theta_2) = \sin \theta_1 \cos \theta_2 \pm \sin \theta_2 \cos \theta_1$
- $\cos(\theta_1 \pm \theta_2) = \cos \theta_1 \cos \theta_2 \mp \sin \theta_1 \sin \theta_2$
- $\tan(\theta_1 \pm \theta_2) = \frac{\tan \theta_1 \pm \tan \theta_2}{1 \mp \tan \theta_1 \tan \theta_2}$