The function \( G(p,t) = \frac{(0.4357p)}{(0.00055724t^2)} \) gives the amount of garbage (in tons) produced in one day by an amusement park when the admission price is \( p \) dollars and the daily high temperature is \( t \) °F.

Use this information to answer the next three questions.

1. Find the amount of garbage produced on a day when the high temperature is 95 degrees Fahrenheit and the admission price is $50.
   
   a. 3.497 tons  
   b. 27.962 tons  
   c. 4.332 tons  
   d. 29.712 tons

2. If 5.3 tons of garbage is produced and the daily high temperature is 75 degrees Fahrenheit, what is the admission price?
   
   a. $105.17  
   b. $105.19  
   c. $38.25  
   d. $38.13

3. Which of the following is the correct interpretation of \( \left. \frac{\partial G}{\partial p} \right|_{(40,80)} = 0.12 \) ?
   
   When the admission price is $40 and the daily high temperature is 80 °F, …
   
   a. …the admission price is increasing by 0.12 dollars per ton of garbage produced.  
   b. …the amount of garbage produced in one day is increasing by 0.12 tons per dollar of admission price.  
   c. …the amount of garbage produced in one day is increasing by 0.12 tons per °F.  
   d. …the amount of garbage produced is increasing by 0.12 °F per dollar of admission price.
Let $B(t, s)$ be the number of minutes it takes for a quart of water to boil when the temperature of the heat source is $t$ hundred °F and the water contains $s$ grams of salt.

Given: $B(3, 2) = 15$  \[ \frac{\partial B}{\partial t} \bigg|_{(3,2)} = -4.3 \quad \frac{\partial B}{\partial s} \bigg|_{(3,2)} = 1.4 \]

4. If the temperature of the heat source changes from 3 hundred to 5 hundred °F, how much additional salt should be added to the quart of water in order for the boiling time to remain 15 minutes? (3 pts)
   a. 0.326 grams
   b. 3.071 grams
   c. 0.651 grams
   d. 6.143 grams

5. At $(1,5)$, will $R(c,t)$ be increasing more rapidly when $c$ increases or when $t$ increases? (2 pts)
   a. when $c$ increases
   b. when $t$ increases

6. Consider the point $(4,3)$. If $c$ increases to 6, approximately by what amount must $t$ change for $R$ (the output) to remain constant? (3 pts)
   a. increase by 5
   b. increase by 2
   c. decrease by 1
   d. decrease by 5
The number of applicants to a small college is given by \( N(t, r) = 400t^{-1}r^{-0.5} \) hundred where \( t \) thousand dollars is the cost of tuition and \( r \) thousand dollars is the cost of room and board.

Check: \( N(15,8) \approx 9.428 \)

Use this context to answer the next five questions.

7. Which of the following notations represents how quickly the number of applicants is changing with respect to the cost of tuition? (1 pt)
   a. \( \frac{\partial N}{\partial t} \)  
   b. \( \frac{dt}{dr} \)  
   c. \( \frac{dr}{dt} \)  
   d. \( \frac{\partial N}{\partial r} \)

8. How quickly is the number of applicants changing with respect to the cost of tuition when tuition is $12,000 and room and board is $5,000? (3 pts)
   a. -0.833 hundred applicants per thousand dollars
   b. -1.200 hundred applicants per thousand dollars
   c. -1.242 hundred applicants per thousand dollars
   d. -1.491 hundred applicants per thousand dollars

9. Two cross-sections of the function \( N(t, r) \) are given in the box shown here.

\[
N(10, r) = \frac{40}{\sqrt{r}} \quad \text{and} \quad N(t, 4) = \frac{200}{t}
\]

Which of the cross-sections from the box above could be used to calculate the number of applicants when tuition is $10,000 and room and board is $4000? (2 pts)
   a. \( N(10, r) \) only  
   b. \( N(t, 4) \) only  
   c. Both  
   d. Neither

10. Which of the cross-sections from the box above could be used to calculate how quickly the number of applicants is changing with respect to tuition when tuition is $10,000 and room and board is $4000? (2 pts)
    a. \( N(10, r) \) only  
    b. \( N(t, 4) \) only  
    c. Both  
    d. Neither

11. Find \( \frac{dN(t, 4)}{dt} \) when \( t = 10 \). (3 pts)
    a. 153.79  
    b. -2  
    c. 20  
    d. 460.52
12. Let \( N(p,s) \) be the number of skiers on a Saturday at a ski resort in Utah when \( p \) dollars is the price of an all-day lift ticket and \( s \) is the number of inches of fresh snow received since the previous Saturday. In general, more snow means more skiers at the resort. Which pair of statements is most likely to be true? 

a. \( \frac{\partial N}{\partial p} < 0 \) and \( \frac{\partial N}{\partial s} < 0 \) 

b. \( \frac{\partial N}{\partial p} > 0 \) and \( \frac{\partial N}{\partial s} < 0 \) 

c. \( \frac{\partial N}{\partial p} > 0 \) and \( \frac{\partial N}{\partial s} > 0 \) 

d. \( \frac{\partial N}{\partial p} < 0 \) and \( \frac{\partial N}{\partial s} > 0 \)

Consider the function \( T(x, y) = x^2 + 2y^2 - 8x + 4y \). Use this function to answer the next two questions.

13. Find the critical point of \( T(x, y) \). 

a. The critical point is at \( T(4, -1) = -18 \). 

b. The critical point is at \( T(0, 0) = 0 \). 

c. The critical point is at \( T(6, 3) = 18 \). 

d. The critical point is at \( T(4, 2) = 0 \). 

14. Classify the critical point of \( T(x, y) \). 

a. The critical point is a relative minimum. 

b. The critical point is a relative maximum. 

c. The critical point is a saddle point. 

d. The critical point cannot be classified with the given information.
15. Find the first partial derivatives of \( f(x, y) = x^y \). (3 pts)

a. \( f_x = (\ln x)x^y \) and \( f_y = yx^{y-1} \)

b. \( f_x = \frac{x^y}{\ln x} \) and \( f_y = \frac{x^{y+1}}{y+1} \)

c. \( f_x = \frac{x^{y+1}}{y+1} \) and \( f_y = \frac{x^y}{\ln x} \)

d. \( f_x = yx^{y-1} \) and \( f_y = (\ln x)x^y \)

Check your Scantron now to make sure it will successfully run. If it does, you will earn one point. (1 pt)

When you are not working on the multiple choice portion of the test, turn your Scantron over so that it cannot be read by others in the room.
RE-READ the directions at the beginning of the test. Then read each question carefully. Provide only one clearly indicated answer to each question. If your answer is illegible, it will be graded as incorrect. Show all work. This portion is 59%.

When possible, set up the specific mathematical notation that is being evaluated to obtain your answer. No credit will be awarded for simply copying generic formulas from the formula sheet.

1. Consider the graph of \( f(x, y) = z \).

\[ f(x, y) = 5x^2y - 3e^x + x^2 \ln(y) \]

\( \frac{\partial f}{\partial x} = 10xy - 3e^x + 2x \ln(y) \)  
Award 1 pt per term

\( \frac{\partial f}{\partial y} = 5x^2 - 0 + \frac{x^2}{y} \)  
Award 1 pt per term

\( f_{xy} = 10x - 0 + \frac{2x}{y} \)  
Award 1 pt per term

Identify and classify the four critical points of \( f(x, y) \) by filling in the blanks below. (8 pts)

- \( f(3, 5) = 700 \) is a saddle point.  
Accept 695 < output value < 705

- \( f(6, 2) = 730 \) is a saddle point.  
Accept 725 < output value < 735

- \( f(6, 5) = 684 \) is a relative minimum.  
Accept 675 ≤ output value ≤ 685

- \( f(3, 2) = 756 \) is a relative maximum.  
Accept 755 ≤ output value ≤ 765

2. Find the following partial derivatives of the function \( f(x, y) = 5x^2y - 3e^x + x^2 \ln(y) \). (9 pts)
The table shows \( M(t, A) \) dollars, the monthly payment required to pay off a loan of \( A \) thousand dollars borrowed at 7% annual interest for \( t \) years.

\[
\begin{array}{c|cccccccc}
A \text{ thousand dollars} & 1 & 5 & 10 & 15 & 20 & 25 & 30 \\
\hline
200 & 17412 & 3981 & 2332 & 1804 & 1555 & 1417 & 1333 \\
175 & 15235 & 3484 & 2041 & 1579 & 1361 & 1240 & 1166 \\
150 & 13059 & 2986 & 1749 & 1353 & 1166 & 1063 & 1000 \\
125 & 10882 & 2488 & 1458 & 1128 & 972 & 886 & 833 \\
100 & 8706 & 1991 & 1166 & 902 & 778 & 708 & 667 \\
75 & 6529 & 1493 & 875 & 677 & 583 & 531 & 500 \\
\end{array}
\]

a. Find the linear cross-sectional model that gives the monthly payment required to pay off \( A \) thousand dollars borrowed at 7% annual interest for 30 years. **Define the model completely.**

\[
M(30, A) = 6.662A + 0.514 \text{ dollars gives the monthly payment to pay off} \\
A \text{ thousand dollars borrowed at 7% annual interest for 30 years, } 75 \leq A \leq 200.
\]

b. Use the **unrounded** model to determine the monthly payment required to pay off a $137,000 loan borrowed at 7% annual interest in 30 years. Round your answer to the nearest cent and include units.

\[
M(30,137) = 913.17
\]

2.5 pts #, \( \frac{1}{2} \) pt units; \( \frac{1}{2} \) pt rounding error;
Up to 2.5 pts partial credit can be awarded if an incorrect answer from part a is used here correctly; Answer must be reasonable based on the given table values.

c. Sketch the SMOOTH $900 contour curve on the table above.
4. A company produces toy favors for children’s parties. Two of its most popular items around Halloween are pirate eye patches and skeleton key chains. The company’s October profit when they produce \( p \) thousand pirate eye patches and \( k \) thousand skeleton key chains can be modeled as

\[
T(p, k) = 1000 - 0.35p^2 + 175p - 0.35pk - 0.525k^2 + 140k
\]
dollars.

Check: \( T(250, 200) = 12375 \)

The first partial derivatives of \( T(p, k) \) are given below:

\[
T_p = -0.7p + 175 - 0.35k \quad \text{and} \quad T_k = -0.35p - 1.05k + 140
\]

a. Set up the system of equations that is used to find the critical point of \( T(p, k) \). (2 pts)

\[
\begin{align*}
T_p &= 0 \quad \text{OR} \quad -0.7p + 175 - 0.35k = 0 \quad \text{OR} \quad -0.7p - 0.35k = -175 \\
T_k &= 0 \quad \text{OR} \quad -0.35p - 1.05k + 140 = 0 \quad \text{OR} \quad -0.35p - 1.05k = -140
\end{align*}
\]

1 pt each equation (A/N)

Exceptions: -\( \frac{1}{2} \) pt for poor notation

Award full credit for algebraically equivalent equations (including matrix equations). If the student explicitly shows that each partial has been set equal to zero but makes a sign error in rewriting the equations, half credit can be awarded.

b. Find the critical point. Show your work (algebraic process or matrices). (3 pts)

\[
\begin{align*}
-0.7p - 0.35k &= -175 \\
-0.35p - 1.05k &= -140
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
-0.7 & -0.35 \\
-0.35 & -1.05
\end{bmatrix}
\begin{bmatrix}
p \\
k
\end{bmatrix}
&= \begin{bmatrix}
-175 \\
-140
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
p \\
k
\end{bmatrix}
&= \begin{bmatrix}
-0.7 & -0.35 \\
-0.35 & -1.05
\end{bmatrix}^{-1}
\begin{bmatrix}
-175 \\
-140
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
p \\
k
\end{bmatrix}
&= \begin{bmatrix}
220 \\
60
\end{bmatrix}
\end{align*}
\]

Award 3 pts for correct work based on part (a)

\( \Rightarrow \) If matrices are used, award 2pts for \([A]\) and 1pt for \([B]\), or 3 pts for the augmented matrix.

- No partial credit for \([A]\) matrix if the order of the coefficients is not consistent.
- No partial credit for \([B]\) if order is inconsistent with the rows of \([A]\).
- No partial credit for \([B]\) if signs are incorrect due to NOT isolating the constants on one side of the equations.
- If errors in \([A]\) or \([B]\) are a result of careless mistakes (copying a number incorrectly, accidentally dropping a negative sign, etc.) partial credit should be awarded (0.5pt for each correct entry).

\( \Rightarrow \) If elimination is used, award 1pt for each step: 1) multiplying one or both equations by the proper constant; 2) adding equations; 3) substituting 1st value to find the 2nd

\( \Rightarrow \) If substitution is used, award 1pt for each correct step: 1) isolating 1st variable; 2) substituting expression into 2nd equation & solving; 3) substituting 1st value to find the 2nd

Deduct \( \frac{1}{2} \) pt for notational errors such as \([A][X]=[B]\) or \([A][B]=[X]\)

Answers: \( p \) & \( k \) must match the key, no exceptions (do not try to follow from incorrect work above)

If \( p \) & \( k \) are incorrect, credit should be given for \( T \) if it follows correctly from the student’s \( p \) & \( k \).

Answer: \( p = 220 \) thousand pirate eye patches \( (1 \text{ pt}) \)

\( k = 60 \) thousand skeleton key chains \( (1 \text{ pt}) \)

\( T = 24,450 \) dollars \( (1 \text{ pt}) \)

Note: This context is continued on the next two pages.
The first partial derivatives of $T(p, k)$ are given below:

\[ T_p = -0.7p + 175 - 0.35k \quad \text{and} \quad T_k = -0.35p - 1.05k + 140 \]

---

c. Find the second partial derivatives for $T(p, k)$. 1 pt each

\[

t_{pp} = -0.7 \quad t_{pk} = -0.35 \quad t_{kp} = -0.35 \quad t_{kk} = -1.05
\]

---

d. What is the value of the determinant of the second partials matrix at the critical point? Do not round your answer.

\[
D(220, 60) = (-0.7)(-1.05) - (-0.35)(-0.35) = 0.735 - 0.1225 = 0.6125
\]

Follow from c; Deduct ½ pt – 1 pt for minor errors if work is shown.

Use the determinant test to classify the critical point of $T(p, k)$ by filling in the blanks below.

The critical point identified in part b is a __________________________

relative maximum

relative maximum or relative minimum or saddle point

because $D(220, 60) = 0.6125$ or $D(220, 60) > 0$ or $0.6125 > 0$ or the determinant at the CP is positive, and

Reason 1: Show the specific notation and value

Reason 2: Show the specific notation and value

(if necessary) $t_{pp}(220, 60) = -0.7$ or $t_{pp}(220, 60) < 0$. (2 pts)

---

e. The $T = 18,000$ contour curve has been sketched for $150 \leq p \leq 350$ and $50 \leq k \leq 200$. Find the missing coordinate for each of the two marked points. Round to three decimal places.

\[
\begin{array}{|c|c|}
\hline
p & k \\
\hline
150 & 181.126 \\
200 & 176.501 \\
250 & 158.562 \\
300 & 126.769 \\
328.174 & 100 \\
350 & 70.489 \\
\hline
\end{array}
\]

2 pts each value, -½ pt to –1 pt for rounding errors depending on the severity
The company’s October profit when they produce \( p \) thousand pirate eye patches and \( k \) thousand skeleton key chains can be modeled as 
\[
T(p, k) = 1000 - 0.35p^2 + 175p - 0.35pk - 0.525k^2 + 140k \quad \text{dollars}.
\]
Check: \( T(250,200) = 12375 \)

The first partial derivatives of \( T(p,k) \) are given below:
\[
T_p = -0.7p + 175 - 0.35k \quad \text{and} \quad T_k = -0.35p - 1.05k + 140
\]

f. Find the slope of the tangent line to the \( T = 18,000 \) contour curve at the point \((300, 126.769)\). 
Round to 3 decimal places. Include units. (4 pts)

\[
\frac{dk}{dp}igr|_{(300, 126.769)} = \frac{-T_p}{T_k}\bigl|_{(300, 126.769)} = \frac{-(79.36915)}{-98.10745} \approx 0.809 \quad \text{thous. key chains} \quad \text{or} \quad \text{key chains}
\]

3.5pts calculation
If answer is correct, award full credit with the exception of a possible 0.5 pt deduction for poor notation (\( \frac{dk}{dp} = -\frac{dp}{dk} \)).
If answer is incorrect, up to 2.5 pts may be awarded for work:
- Up to 2 pts correct ratio such as \( \frac{T_p}{T_k} \), "nDeriv" notation, or numeric values.
- \( \frac{dp}{dk} \) should receive NO credit if it is the only supporting work
- 2 pts was not awarded for the ratio of the input values or \( \frac{T_p}{T_k} \) even if a correct ratio was shown previously
- 0.5 pt negating the ratio

0.5pt units
- 1/2 pt rounding error

g. Use \( \frac{dk}{dp}\bigr|_{(300, 126.769)} \) to estimate the change in \( k \) needed to compensate for \( \Delta p = -10 \) if the output value is to remain the same. Show your work. Interpret the results by completing the sentence below. (5 pts)

\[
\Delta k \approx \Delta p \cdot (-0.809)
\]
Work: 
\[
\approx -10 \cdot (-0.809)
\]
\[
\approx 8.09 \text{ thousand key chains}
\]
Answer: \( \Delta k = 8.09 \)

If the company wants to maintain a profit of $18,000 while decreasing the production of pirate eye patches from 300,000 to 290,000, they will need to increase the production of skeleton key chains from 126,769 to 134,859.