Instructions: You are not permitted to use a calculator on this test. You are not allowed to use a textbook, notes, cell phone, laptop, PDA, or any technology on this test. All devices must be turned off while you are in the testing room.

During this test, any communication with any person (other than the instructor or a designated proctor) in any form, including written, signed, verbal, or digital, is understood to be a violation of academic integrity.

No part of this test may be removed from the examination room.

Read each question very carefully. In order to receive full credit, you must:

1. Show legible, logical, and relevant justification which supports your answer.
2. Use complete and correct mathematical notation.
3. Include proper units, if necessary.
4. Give exact numerical values whenever possible.

You have 90 minutes to complete the entire test.

On my honor, I have neither given nor received inappropriate or unauthorized information at any time before or during this test.

Student’s Signature: ____________________________

<table>
<thead>
<tr>
<th>Free Response Problem</th>
<th>Possible Points</th>
<th>Points Earned</th>
<th>Free Response Problem</th>
<th>Possible Points</th>
<th>Points Earned</th>
</tr>
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<tbody>
<tr>
<td>1.</td>
<td>8</td>
<td></td>
<td>4.a.</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>2.a.</td>
<td>8</td>
<td></td>
<td>4.b.</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>2.b.</td>
<td>8</td>
<td></td>
<td>4.c.</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>2.c.</td>
<td>8</td>
<td></td>
<td>5.</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>2.d.</td>
<td>8</td>
<td></td>
<td>6.</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>8</td>
<td></td>
<td>7.</td>
<td>8</td>
<td></td>
</tr>
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<td></td>
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<td>8.</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>Test Total</td>
<td></td>
<td></td>
<td></td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
Read each question carefully. In order to receive full credit you must show legible, logical, and relevant justification which supports your final answer. Give answers as exact values. You are NOT permitted to use a calculator on any portion of this test.

1. (8 pts.) Use the graph of $f(x)$ to answer the following questions. (1 pt. each)

   Infinite limits should be answered with “$= \infty$” or “$= -\infty$”, whichever is appropriate.

   If the limit does not exist (and cannot be answered as $\infty$ or $-\infty$), state “DNE.”

   a) $\lim_{x \to -5} f(x) = 4$

   b) $\lim_{x \to 5} f(x) = -\infty$

   c) $\lim_{x \to 0} f(x) = -3$

   d) $\lim_{x \to -4} f(x) = DNE$

   e) $\lim_{x \to 3} f(x) = 2$

   f) $\lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = DNE$

   g) $\lim_{x \to 7^-} f(x) = -1$

   h) $\lim_{x \to -6^+} \frac{df}{dx} = 0$
2. (8 pts. each) Find the following limits. Show all work. Do NOT use L’Hôpital’s Rule.

\[
\lim_{x \to -\infty} \left( e^{\ln \left( \frac{7 + \frac{1}{x} + \frac{2}{x^2}}{x^3} \right)} \sec \left( \frac{1}{x^3} \right) \right)
\]

a) \[
= \lim_{x \to -\infty} \left( 7 + \frac{1}{x} + \frac{2}{x^2} \right) \sec \left( \frac{1}{x^3} \right) = \lim_{x \to -\infty} \left( 7 + \frac{1}{x} + \frac{2}{x^2} \right) \lim_{x \to -\infty} \sec \left( \frac{1}{x^3} \right) = (7 + 0 + 0) \sec(0) = 7(1) = 7
\]

Work on Problem:

<table>
<thead>
<tr>
<th>Points Awarded:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finds limit for argument of natural log function.</td>
</tr>
<tr>
<td>Finds limit for argument of secant function</td>
</tr>
<tr>
<td>Simplifies exponential and natural log as inverse functions</td>
</tr>
<tr>
<td>Final answer</td>
</tr>
</tbody>
</table>

Notes:
- Subtract ½ point for missing notation such as: \( x \to a \) (w/o “limit”), omitting \( \lim \) after substitution
- Maximum of 1 point deduction for all notation errors

b) \[
\lim_{t \to 5} \frac{\sqrt{4t+16} - 6}{25t-t^3}
\]

\[
= \lim_{t \to 5} \frac{\sqrt{4t+16} - 6}{25t-t^3} \cdot \frac{\sqrt{4t+16} + 6}{\sqrt{4t+16} + 6} = \lim_{t \to 5} \frac{(4t+16) - 36}{(25t-t^3)(\sqrt{4t+16} + 6)} = \lim_{t \to 5} \frac{4t-20}{t(25-t^3)(\sqrt{4t+16} + 6)}
\]

\[
= \lim_{t \to 5} \frac{4(t-5)}{t(5-t)(5+t)(\sqrt{4t+16} + 6)} = \lim_{t \to 5} \frac{-4}{t(5+t)(\sqrt{4t+16} + 6)} = \frac{-4}{5(5+5)(\sqrt{4(5)+16} + 6)} = \frac{-4}{5(10)(12)} = \frac{1}{150}
\]

Work on Problem:

<table>
<thead>
<tr>
<th>Points Awarded:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recognizes implicitly or explicitly indeterminate form 0/0.</td>
</tr>
<tr>
<td>Uses conjugate to rewrite</td>
</tr>
<tr>
<td>Algebra to get a reduced form for which substitution works</td>
</tr>
<tr>
<td>Correctly evaluates limit.</td>
</tr>
</tbody>
</table>

Notes:
- Subtract ½ point for the untrue statement: \( \lim_{x \to a} \frac{f(x)}{g(x)} = 0 \).
- Subtract 1 point for not presenting the correct reduced form before substitution
- Maximum of 1 point deduction for all notation errors
c) 
\[
\lim_{x \to 4} \sin \left( \cos^{-1} \left( \ln e^{\left(\frac{x^2-1}{30}\right)} \right) \right)
\]
\[
= \sin \left( \cos^{-1} \left( \ln e^{\left(\frac{4^2-1}{30}\right)} \right) \right) = \sin \left( \cos^{-1} \left( \ln \left(\frac{e^{15}}{30}\right) \right) \right) = \sin \left( \cos^{-1} \left( \frac{15}{30} \right) \right) = \sin \left( \cos^{-1} \left( \frac{1}{2} \right) \right) = \sin \left( \frac{\pi}{3} \right) = \frac{\sqrt{3}}{2}
\]

<table>
<thead>
<tr>
<th>Work on Problem:</th>
<th>Points Awarded:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Substitutes.</td>
<td>1 point</td>
</tr>
<tr>
<td>Simplifies exponential and natural log as inverse functions</td>
<td>2 points</td>
</tr>
<tr>
<td>Finds $\cos^{-1}(1/2)$</td>
<td>3 points</td>
</tr>
<tr>
<td>Finds $\sin(\pi/3)$</td>
<td>2 points</td>
</tr>
</tbody>
</table>

Notes:
- Subtract ½ point for missing notation such as: $x \to a$ (w/o “limit”), omitting $=$, including $\lim_{x \to a}$ after substitution
- Maximum of 1 point deduction for all notation errors
d) 
\[
\lim_{x \to -\infty} \left( \frac{\sqrt{16x^4 + 64x^2 + x^2}}{2x^2 - \pi^5} \right)
\]

\[
= \lim_{x \to -\infty} \frac{\sqrt{x^4 \left( 16 \frac{64}{x^2} \right) + x^2}}{2x^2 - \pi^5} = \lim_{x \to -\infty} \frac{x^2 \sqrt{16 + 64 \frac{1}{x^2}} + x^2}{2x^2 - \pi^5} = \lim_{x \to -\infty} \frac{x^2}{x^2} \sqrt{16 + 64 \frac{1}{x^2}} + \frac{x^2}{x^2}
\]

\[
= \lim_{x \to -\infty} \frac{\sqrt{16 \frac{64}{x^2}} + 1}{2 - \frac{\pi^5}{x^2}} = \sqrt{16 + 0} + 1 \quad \frac{2 - 0}{2} = 4 + 1 \quad 2 = \frac{5}{2}
\]

**Work on Problem:**

<table>
<thead>
<tr>
<th>Points</th>
<th>Work on Problem</th>
</tr>
</thead>
</table>
| 1 point | Recognizes implicitly or explicitly indeterminate form \( \frac{\infty}{\infty} \).
| 5 points | Rewrites to get a form for which substitution works.
| 2 points | Correctly evaluates limit.

**Notes:**

- Award 3 points total for recognizing \( \frac{\infty}{\infty} \) (I.F.) and the need to rewrite, but incorrect algebra leads to incorrect answer or leads to incorrect work to arrive at answer.
- Subtract ½ point for the untrue statement: \( \lim_{x \to -\infty} \frac{f(x)}{g(x)} = \infty \)
- Subtract ½ point for missing notation such as: \( x \to a \) (w/o “limit”), omitting =, including \( \lim_{x \to a} \)
- Maximum of 1 point deduction for all notation errors.
3. (8 pts.) Let \( f(x) = \frac{2x^2 - 9x - 5}{x - 5} \). Use the epsilon-delta definition of a limit to prove
\[
\lim_{{x \to 5}} f(x) = 11.
\]
we want
\[
|f(x) - 11| < \varepsilon
\]
\[
\left| \frac{2x^2 - 9x - 5}{x - 5} - 11 \right| < \varepsilon
\]
\[
\left| \frac{(2x + 1)(x - 5)}{x - 5} - 11 \right| < \varepsilon
\]
\[
|2(x + 1) - 11| < \varepsilon
\]
\[
|2(x - 5)| < \varepsilon
\]
\[
|x - 5| < \frac{\varepsilon}{2} \Rightarrow \text{choose } \delta = \frac{\varepsilon}{2}
\]

Proof:

Let \( \varepsilon > 0 \) be given.

Choose \( \delta = \frac{\varepsilon}{2} \Rightarrow 0 < |x - 5| < \delta = \frac{\varepsilon}{2} \). Then
\[
|f(x) - 11| = \left| \frac{2x^2 - 9x - 5}{x - 5} - 11 \right| = \left| \frac{(2x + 1)(x - 5)}{x - 5} - 11 \right| = |2x + 1 - 11| = |2x - 10| = 2|x - 5| < 2\delta = 2\left( \frac{\varepsilon}{2} \right) = \varepsilon.
\]

**Work on Problem:**

<table>
<thead>
<tr>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Determines a value for ( \delta )</td>
</tr>
<tr>
<td>Proof</td>
</tr>
</tbody>
</table>

**Notes:**

- Subtract 1pt. if epsilon and delta were switched but the student was consistent throughout. At least 2 points were taken off if they were switched halfway through the problem.
- Subtract 1pt. if '=' was used instead of '<' in finding a value for delta.
- Subtract 1pt. if limit notation was used.
- Subtract ½ point if the absolute values were missing more than once.
- Subtract ½ if "let epsilon > 0 be given" (or something similar) was missing from the proof.
4. (7 pts. each) Find the derivatives of the following functions. Assume \( g(x) \) is a differentiable function wherever it appears. Do NOT simplify your answers.

a) 
\[
f(x) = \frac{1 + xe^x}{1 + e^x}
\]
\[
f'(x) = \frac{(1 + e^x)(xe^x + e^x) - (1 + xe^x)e^x}{(1 + e^x)^2}
\]

b) 
\[
f(x) = \frac{(x^2 + 1)g(x)}{x^3}
\]
\[
f'(x) = \frac{(x^3)[(x^2 + 1)g'(x) + g(x)(2x)] - (x^2 + 1)g(x)(3x^2)}{(x^3)^2}
\]

<table>
<thead>
<tr>
<th>Work on Problem:</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Applies Quotient Rule correctly</td>
<td>4 points</td>
</tr>
<tr>
<td>Applies Product Rule correctly</td>
<td>3 point</td>
</tr>
</tbody>
</table>

Notes: 

- 

<table>
<thead>
<tr>
<th>Work on Problem:</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Derivative of ( \sqrt{t} )</td>
<td>3 points</td>
</tr>
<tr>
<td>Derivative of two constants (1 point each)</td>
<td>2 points</td>
</tr>
<tr>
<td>Constant multiple</td>
<td>2 point</td>
</tr>
</tbody>
</table>

Notes: 

-
5. (7 pts) Let \( h(x) = \frac{-2f(x)}{g(x)} \).

Find \( h'(1) \) if \( f(1) = -3 \), \( g(1) = 4 \), \( f'(1) = -2 \), and \( g'(1) = 7 \).

\[
h(x) = \frac{-2f(x)}{g(x)}
\]

\[
h'(x) = (-2) \left[ \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \right]
\]

\[
h'(1) = (-2) \left[ \frac{g(1)f'(1) - f(1)g'(1)}{[g(1)]^2} \right]
\]

\[
h'(1) = (-2) \left[ \frac{(4)(-2) - (-3)(7)}{[4]^2} \right]
\]

\[
h'(1) = (-2) \left[ \frac{-8 - (-21)}{[4]^2} \right]
\]

\[
h'(1) = (-2) \left[ \frac{13}{16} \right]
\]

\[
h'(1) = -\frac{13}{8}
\]

**Work on Problem:**

<table>
<thead>
<tr>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Applies Quotient Rule correctly</td>
</tr>
<tr>
<td>Substitutes given values</td>
</tr>
<tr>
<td>Final answer</td>
</tr>
</tbody>
</table>

**Notes:**

-
6. (9 pts.) Let \( f(x) = 3x^3 - 13x \).

a) (3 pts.) Find all values of \( x \) such that the line tangent to \( f(x) \) is parallel to the line \( y = 23x + 1 \).

\[
\begin{align*}
  f(x) &= 3x^3 - 13x \\
  f'(x) &= 9x^2 - 13 \\
  f'(1) &= 9(1)^2 - 13 = -4 \\
  \Rightarrow m_{\text{normal}} &= \frac{1}{4}
\end{align*}
\]

Work on Problem:

<table>
<thead>
<tr>
<th>Points</th>
<th>Derivative of ( f )</th>
<th>Sets ( f' ) equal to 23</th>
<th>Two solutions (1/2 each)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 point</td>
<td>1 point</td>
<td>1 point</td>
<td></td>
</tr>
</tbody>
</table>

Notes:

- 

b) (3 pts.) Find the equation of the line normal (perpendicular) to \( f(x) \) at \( x = 1 \). Put your final answer in slope-intercept form (\( y = mx + b \)).

\[
\begin{align*}
  f(x) &= 3x^3 - 13x \\
  f(1) &= 10 \\
  \Rightarrow & \text{ point(1, -10)} \\
  f'(1) &= -4 \\
  y - (-10) &= \frac{1}{4} (x-1) \\
  y &= \frac{1}{4} x - \frac{1}{4} - 10 \\
  y &= \frac{1}{4} x - \frac{41}{4}
\end{align*}
\]

Work on Problem:

<table>
<thead>
<tr>
<th>Points</th>
<th>Calculates ( f'(1) )</th>
<th>Calculates ( f(1) )</th>
<th>Determines slope of normal</th>
<th>Equation of normal line</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2 point</td>
<td>1/2 point</td>
<td>1 point</td>
<td>1 point</td>
<td></td>
</tr>
</tbody>
</table>

Notes:

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c) (3 pts.) Let \( g(x) = -10x + 7 \). Show that there exists a solution to the equation \( f(x) = g(x) \).

\[
\begin{align*}
  f(x) &= g(x) \\
  3x^3 - 13x &= -10x + 7 \\
  3x^3 - 3x - 7 &= 0 \\
  h(x) &= 3x^3 - 3x - 7 \\
  h(x) \text{ is continuous on its domain } (-\infty, \infty). \\
  h(0) &= -7 < 0 \\
  h(2) &= 11 > 0 \\
  \Rightarrow & \text{ by IVT there exists some } c \in (0, 2) \text{ such that } h(c) = 0 \\
  \Rightarrow & h \text{ has an } x \text{-intercept at } x = c \\
  \Rightarrow & f(x) = g(x) \text{ has a solution at } x = c.
\end{align*}
\]

Work on Problem:

<table>
<thead>
<tr>
<th>Points</th>
<th>Establishes continuity</th>
<th>Finds an ( x ) such that ( f(x) &lt; 0 )</th>
<th>Finds an ( x ) such that ( f(x) &gt; 0 )</th>
<th>Mentions IVT</th>
<th>Conclusion that ( f(x) = g(x) ) has a solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2 point</td>
<td>1/2 point</td>
<td>1/2 point</td>
<td>1/2 point</td>
<td>1/2 point</td>
<td>1/2 point</td>
</tr>
</tbody>
</table>

Notes:

-
7. (8 pts.) Use the **limit definition** of the derivative to find \( f'(x) \) if \( f(x) = \frac{x}{2x+1} \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
= \lim_{h \to 0} \frac{x + h}{2(x + h) + 1} - \frac{x}{2x + 1} \\
= \lim_{h \to 0} \frac{x + h}{2x + 2h + 1} - \frac{x}{2x + 1} \\
= \lim_{h \to 0} \frac{(x + h)(2x + 1) - x(2x + 2h + 1)}{(2x + 2h + 1)(2x + 1)} \\
= \lim_{h \to 0} \frac{2x^2 + x + 2xh + h - 2x^2 - 2xh - x}{h(2x + 2h + 1)(2x + 1)} \\
= \lim_{h \to 0} \frac{h}{h(2x + 2h + 1)(2x + 1)} \\
= \lim_{h \to 0} \frac{1}{(2x + 2h + 1)(2x + 1)} \\
= \frac{1}{(2x+1)^2}
\]

**Work on Problem**

<table>
<thead>
<tr>
<th>Points</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 Points</td>
<td>States the formula for the limit definition of the derivative. Okay if implied</td>
</tr>
<tr>
<td>2 Points</td>
<td>Substitutes ((x + h)) correctly into the definition</td>
</tr>
<tr>
<td>3 Points</td>
<td>Simplifies to a form where direct substitution works</td>
</tr>
<tr>
<td>1 Point</td>
<td>(\bullet) 1 point for getting a common denominator for ( f(x+h) ) and ( f(x) )</td>
</tr>
<tr>
<td></td>
<td>(\bullet) 1 point for expanding the polynomials in the numerator after combining ( f(x+h) ) and ( f(x) ) into a single function</td>
</tr>
<tr>
<td></td>
<td>(\bullet) 1 point for simplifying terms to the point where the limit can be evaluated using direct substitution</td>
</tr>
<tr>
<td>1 Point</td>
<td>Correctly evaluates limit (no credit if this doesn’t follow from work)</td>
</tr>
</tbody>
</table>

**Notes:**
- Subtract 8 points for not using the limit definition of the derivative
- Subtract 2 points if no work is shown between simplifying from getting a common denominator and taking the limit
- Max of 2 points total for all notation errors
- Subtract 1 point if \( \lim_{h \to 0} \) is missing at any step where it should be present
- Subtract \(\frac{1}{2}\) point if \( \lim_{h \to 0} \) is written after direct substitution (after the limit has been taken)
- Work is only followed after a mistake if the mistake does not reduce the difficulty of the problem
8. (7 pts.) Let \( f(x) \) be a function such that the following inequality is true for all real numbers \( a \) and \( b \). Find \( \lim_{x \to a} f(x) \).

\[
b - |x-a| \leq 2f(x) \leq b + |x-a|.
\]

\[
\begin{align*}
\lim_{x \to a} (b - |x-a|) &= b - |a-a| \\
&= b - 0 \\
&= b \\
\lim_{x \to a} (b + |x-a|) &= b + |a-a| \\
&= b + 0 \\
&= b
\end{align*}
\]

By the Squeeze Theorem
\[
\lim_{x \to a} 2f(x) = b \\
\Rightarrow 2 \lim_{x \to a} f(x) = b \\
\Rightarrow \lim_{x \to a} f(x) = \frac{b}{2}
\]