Instructions: You are not permitted to use a calculator on any portion of this test. You are not allowed to use a textbook, notes, cell phone, laptop, PDA, or any technology on any portion of this test. All devices must be turned off while you are in the testing room.

During this test, any communication with any person (other than the instructor or a designated proctor) in any form, including written, signed, verbal, or digital, is understood to be a violation of academic integrity.

No part of this test may be removed from the examination room.

Read each question very carefully. In order to receive full credit for the free response portion of the test, you must:
1. Show legible and logical (relevant) justification which supports your final answer.
2. Use complete and correct mathematical notation.
3. Include proper units, if necessary.
4. Give exact numerical values whenever possible.

You have 90 minutes to complete the entire test.

On my honor, I have neither given nor received inappropriate or unauthorized information at any time before or during this test.

Student’s Signature: ________________________________________________
Read each question carefully. In order to receive full credit you must show legible and logical (relevant) justification which supports your final answer. Give answers as exact values. You are NOT permitted to use a calculator on any portion of this test.

1. (10 pts. each) Find the indicated limits, noting L’Hôpital’s Rule wherever it is used.

a) \( \lim_{x \to 0^+} \left( 1 + \frac{2}{x} \right)^x \)

\[
\begin{align*}
\lim_{x \to 0^+} \left( 1 + \frac{2}{x} \right)^x &= \lim_{x \to 0^+} e^{\ln \left( 1 + \frac{2}{x} \right)^x} \\
&= \lim_{x \to 0^+} e^{x \ln \left( 1 + \frac{2}{x} \right)} \\
&= \lim_{x \to 0^+} \left( 1 + \frac{2}{x} \right)^{\frac{x}{x}} \\
&= \lim_{x \to 0^+} \left( \frac{1 + \frac{2}{x}}{1} \right)^{\frac{x}{x}} \\
&= \lim_{x \to 0^+} \left( 1 + \frac{2}{x} \right)^{\frac{x}{1}} \\
&= \lim_{x \to 0^+} \left( 1 + \frac{2}{x} \right)^x \\
&= e^{\lim_{x \to 0^+} \frac{2}{x}} \\
&= e^0 \\
&= 1
\end{align*}
\]

<table>
<thead>
<tr>
<th>Work on Problem:</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recognizes indeterminate form (explicitly or implicitly)</td>
<td>1 point</td>
</tr>
<tr>
<td>Rewrites using ( e ) and natural log</td>
<td>1 point</td>
</tr>
<tr>
<td>Applies log property</td>
<td>1 point</td>
</tr>
<tr>
<td>Recognizes ( 0 \cdot \infty )</td>
<td>1 point</td>
</tr>
<tr>
<td>Divides by reciprocal</td>
<td>2 points</td>
</tr>
<tr>
<td>Applies L’Hopital’s Rule correctly</td>
<td>2 points</td>
</tr>
<tr>
<td>Evaluates the resulting limit</td>
<td>1 point</td>
</tr>
<tr>
<td>Final Answer</td>
<td>1 point</td>
</tr>
</tbody>
</table>

Notes:
- Subtract \( \frac{1}{2} \) point for failing to indicate use of L’Hopital’s Rule
- Subtract \( \frac{1}{2} \) point for each notation error with a maximum of one point total for all notation errors (excluding errors indicating use of L’Hopital’s Rule)
- Subtract \( \frac{1}{2} \) point for statement: anything = an indeterminate form
- Subtract \( \frac{1}{2} \) point for wrong indeterminate form
b) \[ \lim_{{x \to 0^+}} \frac{e^{kx} - 1 - kx}{x^2} \quad (k \text{ a constant}) \]

\[
\begin{align*}
\lim_{{x \to 0^+}} \frac{e^{kx} - 1 - kx}{x^2} & = 0 \quad \text{, I. F.} \\
L & = \lim_{{x \to 0^+}} \frac{k e^{kx} - k}{2x} \\
& = \lim_{{x \to 0^+}} \frac{k^2 e^{kx}}{2} \\
& = \frac{k^2 e^{k(0)}}{2} \\
& = \frac{k^2}{2}
\end{align*}
\]

**Work on Problem:**

<table>
<thead>
<tr>
<th>Work on Problem:</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Recognizes indeterminate form (explicitly or implicitly)</td>
<td>1 point</td>
</tr>
<tr>
<td>Applies L’Hopital’s Rule correctly</td>
<td>3 point</td>
</tr>
<tr>
<td>Recognizes indeterminate form (explicitly or implicitly)</td>
<td>1 point</td>
</tr>
<tr>
<td>Applies L’Hopital’s Rule correctly</td>
<td>3 points</td>
</tr>
<tr>
<td>Evaluates the resulting limit</td>
<td>1 point</td>
</tr>
<tr>
<td>Final Answer</td>
<td>1 point</td>
</tr>
</tbody>
</table>

**Notes:**
- Subtract \( \frac{1}{2} \) point for failing to indicate use of L’Hopital’s Rule
- Subtract \( \frac{1}{2} \) point for each notation error with a maximum of one point total for all notation errors (excluding errors indicating use of L’Hopital’s Rule)
- Subtract \( \frac{1}{2} \) point for statement: anything = an indeterminate form
2. (10 pts.) Find the absolute maximum and minimum values of \( f(x) \) on the given interval. Put your final answers in the appropriate spaces at the bottom of the page.

\[
f(x) = 125x^2(1-x)^3, \quad 0 \leq x \leq 2
\]

\[
f(x) = 125x^2(1-x)^3, \quad 0 \leq x \leq 2
\]

\[
f'(x) = 125 \left[ x^2(3)(1-x)^2(-1) + (1-x)^3(2x) \right]
\]

\[
= 125 \left[ x^2(3)(1-x)^2(-1) + (1-x)^3(2x) \right]
\]

\[
= 125 \left[ x(1-x)^2 \left( x(-3) + (1-x)(2) \right) \right]
\]

\[
= 125x(1-x)^2 \left( -3x + 2 - 2x \right)
\]

\[
= 125x(1-x)^2 \left( 2 - 5x \right)
\]

Solve \( f'(x) = 0 \) to find critical points

\[
0 = 125x(1-x)^2 \left( 2 - 5x \right)
\]

\[
x = 0, \quad \frac{2}{5}, \quad 1
\]

Evaluate \( f(x) \) at end points and critical points

\[
f(0) = 0
\]

\[
f \left( \frac{2}{5} \right) = 125 \left( \frac{2}{5} \right)^2 \left( 1 - \frac{2}{5} \right)^3 = 125 \left( \frac{2}{5} \right)^2 \left( \frac{3}{5} \right)^3 = 125 \left( \frac{4}{25} \right) \left( \frac{27}{125} \right) = \frac{108}{25}
\]

\[
f(1) = 0
\]

\[
f(2) = 125 \left( 2 \right)^3 \left( -1 \right)^3 = 125(4)(-1) = -500
\]

Absolute max of \( \frac{108}{25} \) at \( x = \frac{2}{5} \); Absolute min of \( -500 \) at \( x = 2 \)

<table>
<thead>
<tr>
<th>Work on Problem:</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finds ( f'(x) )</td>
<td>3 points</td>
</tr>
<tr>
<td>Finds the critical values</td>
<td>3 points</td>
</tr>
<tr>
<td>Function values at the endpoints and the two critical points</td>
<td>2 points</td>
</tr>
<tr>
<td>States absolute extremes (1 point each)</td>
<td>2 points</td>
</tr>
</tbody>
</table>

**Notes:**
- Subtract \( \frac{1}{2} \) point for missing or incorrect derivative notation.
- Subtract 1 point for wrong verification technique based on domain
- Subtract \( \frac{1}{2} \) point for missing units
3. Let \( f(x) = x^2 e^x \)

a) **(7 pts.)** Determine the **equation** of any horizontal asymptotes on the graph of \( f(x) \).

\[
\lim_{x \to \infty} x^2 e^x = \infty \Rightarrow \text{No H.A.}
\]

\[
\lim_{x \to -\infty} x^2 e^x = \infty \cdot 0, \text{ I.F.}
\]

\[
= \lim_{x \to -\infty} \frac{x^2}{e^{-x}} = \infty, \text{ I.F.}
\]

\[
L = \lim_{x \to -\infty} \frac{2x}{e^{-x}} = -\infty, \text{ I.F.}
\]

\[
L = \lim_{x \to -\infty} \frac{2}{e^{-x}} = 0
\]

\[\Rightarrow y = 0 \text{ is H.A.}\]

**Work on Problem:**

<table>
<thead>
<tr>
<th>Points</th>
<th>Limit at ( \infty )</th>
<th>considering the limit at negative infinity of ( x^2 e^x )</th>
<th>re-writing it as the limit of a quotient</th>
<th>first application of L'Hospital</th>
<th>second application of L'Hospital</th>
<th>evaluating limit as 0</th>
<th>stating horizontal asymptote is ( y = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 point</td>
<td>1 point</td>
<td>1 point</td>
<td>1 point</td>
<td>1 point</td>
<td>1 point</td>
<td>1 point</td>
<td>1 point</td>
</tr>
</tbody>
</table>

**Notes:**
- Subtract \( \frac{1}{2} \) point for failing to indicate use of L’Hôpital’s Rule
- Subtract \( \frac{1}{2} \) point for each notation error with a maximum of one point total for all notation errors (excluding errors indicating use of L’Hôpital’s Rule)
- Subtract \( \frac{1}{2} \) point for statement: anything = an indeterminate form

b) **(7 pts.)** Determine the intervals on which \( f(x) \) is increasing or decreasing. Be sure to show the calculation of the first derivative. Put your final answers in the appropriate spaces below.

\[
f(x) = x^2 e^x
\]

\[
f'(x) = x^2 e^x + 2xe^x
\]

\[
f''(x) = xe^x(x + 2)
\]

Find critical values.

\[
xe^x(x + 2) = 0
\]

\[
x = 0 \text{ or } x = -2
\]

Increasing: \((-\infty, -2)\), \((0, \infty)\) Decreasing: \((-2, 0)\)

**Work on Problem:**

<table>
<thead>
<tr>
<th>Points</th>
<th>first derivative (no partial credit)</th>
<th>setting ( f'(x) = 0 )</th>
<th>( x = 0 ), ( x = 2 )</th>
<th>increasing intervals (0.5 pts each)</th>
<th>decreasing interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 points</td>
<td>1 point</td>
<td>1 point</td>
<td>2 points</td>
<td>1 point</td>
<td>1 point</td>
</tr>
</tbody>
</table>

**Notes:**
- 
c) (7 pts.) The second derivative of $f(x)$ is shown below. Determine the intervals on which $f(x)$ is concave up or concave down. Put your final answers in the appropriate spaces below.

$$f''(x) = e^x(x^2 + 4x + 2)$$

Solve $f''(x) = 0$

$$e^x(x^2 + 4x + 2) = 0$$

$$x^2 + 4x + 2 = 0$$

Solving with quadratic formula

$$x = -2 \pm \sqrt{2}$$

Concave Up: $(-\infty, -2 - \sqrt{2})$, $(-2 + \sqrt{2}, \infty)$

Concave Down: $(-2 - \sqrt{2}, -2 + \sqrt{2})$

Work on Problem:

<table>
<thead>
<tr>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>setting $f''(x) = 0$</td>
</tr>
<tr>
<td>solving with quadratic formula</td>
</tr>
<tr>
<td>concave up intervals (0.5 pts each)</td>
</tr>
<tr>
<td>concave down intervals</td>
</tr>
</tbody>
</table>

Notes:

- 

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d) (8 pts.) Sketch $f(x)$. Show the $x$-values at each point where $f$ has a local extreme or inflection point. You do not need to show the $y$-values at these points.

\[
\begin{align*}
\text{Work on Problem:} & \quad \text{Points} \\
\text{labels of min/max/IP} & \quad 1 \text{ point} \\
\text{horizontal asymptote } y = 0 (\text{or follows work from (a)}) & \quad 1 \text{ point} \\
\text{going to infinity as } x \text{ goes to infinity (or follows work from (a))} & \quad 1 \text{ point} \\
\text{follows increasing/decreasing intervals from (b)} & \quad 2.5 \text{ points} \\
\text{follows concave up/concave down intervals from (c)} & \quad 2.5 \text{ points} \\
\end{align*}
\]

Notes:

- 

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4. (12 pts.) Consider a flat-bottomed circular cylinder of radius 4 inches. You place a steel marble of radius \( r \) \((0 < r < 4)\) at the bottom of the cylinder and add water to the cylinder until the marble is submerged (until the level of water reaches the top of the marble; see plots). Find the radius of the marble that maximizes the amount of water needed to cover it.

Note: The volume \( V_c \) of a right circular cylinder of radius \( R \) and height \( h \) is \( V_c = \pi R^2 h \)

Note: The volume \( V_s \) of a sphere of radius \( r \) is \( V_s = \frac{4\pi}{3} r^3 \)

Let \( h \) be the height of water in the cylinder
When the marble is submerged, \( h = 2r \)

Let \( V \) be the volume of water in the cylinder
\[
V = V_c - V_s
\]
\[
V = \pi (4)^2 h - \frac{4\pi}{3} r^3
\]
\[
= \pi (4)^2 (2r) - \frac{4\pi}{3} r^3
\]
\[
= 32\pi r - \frac{4\pi}{3} r^3
\]
\[
dV \over dr = 32\pi - 4\pi r^2
\]
Find critical values
\[
0 = 32\pi - 4\pi r^2
\]
\[
r^2 = 8
\]
\[
r = \sqrt{8} \quad \text{(omit } r = -\sqrt{8})
\]

<table>
<thead>
<tr>
<th>Work on Problem:</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recognize ( R=4 )</td>
<td>1.5 points</td>
</tr>
<tr>
<td>Recognize ( h=3r )</td>
<td>1.5 points</td>
</tr>
<tr>
<td>Volume of water function in terms of ( r )</td>
<td>3 points</td>
</tr>
<tr>
<td>Derivative of volume</td>
<td>2 points</td>
</tr>
<tr>
<td>Set derivative equal to zero</td>
<td>1 point</td>
</tr>
<tr>
<td>Solves for ( r )</td>
<td>1.5 points</td>
</tr>
<tr>
<td>Verify max</td>
<td>1 point</td>
</tr>
<tr>
<td>Units</td>
<td>0.5 points</td>
</tr>
</tbody>
</table>

Notes:
- Subtract \( \frac{1}{2} \) point for notation errors
- Subtract \( \frac{1}{2} \) point if verify with closed interval
5. (10 pts.) Find the position function $s(t)$ for an object moving along the $x$-axis with velocity $v(t) = \sin t - \cos t$ and $s\left(\frac{3\pi}{2}\right) = -1$.

$$s(t) = -\cos t - \sin t + C$$

Apply $s\left(\frac{3\pi}{2}\right) = -1$

$$-1 = -\cos\left(\frac{3\pi}{2}\right) - \sin\left(\frac{3\pi}{2}\right) + C$$

$$-1 = 0 - (-1) + C$$

$$-1 = C$$

$s(t) = -\cos t - \sin t - 2$

<table>
<thead>
<tr>
<th>Work on Problem</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finds general antiderivative of $v(t)$</td>
<td>5 points</td>
</tr>
<tr>
<td>Applies initial condition</td>
<td>2 points</td>
</tr>
<tr>
<td>Solves for constant of integration</td>
<td>2 points</td>
</tr>
<tr>
<td>States final $s(t)$ with constant</td>
<td>1 point</td>
</tr>
</tbody>
</table>

Notes:

6. (7 pts.) The graph of $f'(x)$ is shown below. Use it to sketch a graph of $f(x)$ on the interval $[0, 4]$ if $f(0) = -1$ and $f(x)$ is continuous on the interval $[0, 4]$.

<table>
<thead>
<tr>
<th>Work on Problem</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graph starts at (0, -1)</td>
<td>2 points</td>
</tr>
<tr>
<td>Correct slope on three intervals</td>
<td>3 points</td>
</tr>
<tr>
<td>Continuous</td>
<td>1 point</td>
</tr>
<tr>
<td>Line segments with corners</td>
<td>1 point</td>
</tr>
</tbody>
</table>

Notes:
7. **(12 pts.)** Compute the definite integral \( \int_0^2 (x - x^3) \, dx \) as the limit of a right Riemann sum.

   a. **(1 pt.)** Let \( n \) be the number of equal-width subintervals into which the interval [0, 2] is divided. State \( \Delta x \), the width of each subinterval, in terms of \( n \).

   \[
   \Delta x = \frac{2 - 0}{n} = \frac{2}{n}
   \]

   **Work on Problem:**
<table>
<thead>
<tr>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finds ( \Delta x ).</td>
</tr>
</tbody>
</table>
   **Notes:**
   * Grade all or nothing

   b. **(1 pt.)** Find an expression for the right endpoint of the \( i \)-th subinterval, \( x_i \).

   \[
   x_i = 0 + i(\Delta x) = i \left( \frac{2}{n} \right) = \frac{2i}{n}
   \]

   **Work on Problem:**
<table>
<thead>
<tr>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finds ( x_i ).</td>
</tr>
</tbody>
</table>
   **Notes:**
   * Grade all or nothing. Simplification not required

   c. **(2 pts.)** Find an expression for the height of the \( i \)-th rectangle at the right endpoint of the \( i \)-th subinterval. The expression should be in terms of \( i \) and \( n \).

   \[
   \text{height} = f(x_i) = \frac{2i}{n} - \left( \frac{2i}{n} \right)^3
   \]

   **Work on Problem:**
<table>
<thead>
<tr>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Substitutes ( x_i ) into the integrand.</td>
</tr>
</tbody>
</table>
   **Notes:**
   * Simplification not required
   * Answer must be in terms of terms of \( i \) and \( n \).
   * Can give full credit for substituting an incorrect \( x_i \) into the integrand
d. (2 pts.) Find an expression for the area of the $i$-th rectangle. The expression should be in terms of $i$ and $n$.

$$\text{Area} = f(x_i)\Delta x = \left[\frac{2i}{n} - \left(\frac{2i}{n}\right)^3\right] \frac{2}{n}$$

<table>
<thead>
<tr>
<th>Work on Problem:</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Multiplies height of rectangle (result of part c) by width (result of part a).</td>
<td>2 points</td>
</tr>
</tbody>
</table>

Notes:
- Simplification not required.
- Award 2 points for multiplication of results form part (a) and part (c) even if incorrect.

e. (3 pts.) Use the result of Part (d) to find a formula for the sum of the areas of the $n$ rectangles. Simplify so that your final answer is in terms of $n$ only.

$$\sum_{i=1}^{n} c = cn, \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{i=1}^{n} \left(\frac{4i}{n^2} - \frac{16i^3}{n^4}\right) = \frac{4n}{n^2} - \frac{n^3}{n^4} - \frac{16n^3}{n^4} - \frac{n^3}{n^4} = \frac{2}{n^2} - \frac{n^2(n+1)^2}{n^4}$$

$$= \frac{2(n+1)}{n} - \frac{4(n+1)^2}{n^2}$$

<table>
<thead>
<tr>
<th>Work on Problem:</th>
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<tbody>
<tr>
<td>Sets up the sum.</td>
<td>1 point</td>
</tr>
<tr>
<td>Uses the algebraic properties to distribute the sum.</td>
<td>1 point</td>
</tr>
<tr>
<td>Uses summation formulas to get an answer in terms of $n$</td>
<td>1 point</td>
</tr>
</tbody>
</table>

Notes:
- Simplification not required.
- Award 2 points for correct work using an incorrect result from part (d).

f. (3 pts.) Find the exact value of $\int_0^2 (x - x^3) \, dx$ by taking a limit of your result in (e).

$$\int_0^2 (x - x^3) \, dx = \lim_{n \to \infty} \left(\frac{2(n+1)}{n} - \frac{4(n+1)^2}{n^2}\right) = 2 \lim_{n \to \infty} \frac{(n+1)}{n} - 4 \lim_{n \to \infty} \frac{(n+1)^2}{n^2} = 2(1) - 4(1) = -2$$

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<tr>
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<tbody>
<tr>
<td>Sets up the limit as $n \to \infty$ for any result from part (d).</td>
<td>1 point</td>
</tr>
<tr>
<td>Evaluates the limit.</td>
<td>2 points</td>
</tr>
</tbody>
</table>

Notes:
- Restatement of the definite integral is not required.
- Students may state the answer for this limit without showing work.
- Subtract a maximum of one point for notation errors.
- Award 2 points for correct work using an incorrect result from part (e).