Morning Exam - 2019 CCC - Version A

1. Suppose that $f(x)$ is a twice differentiable function such that $f(2) = 10$ and $f'(x) = x^2 f(x)$. Find $f''(2)$.

   (A) $\frac{5}{4}$ (B) 200 (C) 0 (D) $-\frac{15}{8}$ (E) none of these

2. Find $\lim_{x \to 0} \frac{\sin(3x)\sin(2x)}{x \tan(6x)}$.

   (A) 0 (B) Does not exist (C) 1 (D) $\frac{5}{6}$ (E) none of these

3. The tangent line(s) to $y = x^2$ that pass through the point $(3, -7)$ have slope(s).

   (A) -2 and 14 (B) -1 and 4 (C) -2 only (D) -1 only (E) none of these

4. The floor function $\lfloor x \rfloor$ gives the greatest integer that is smaller than or equal to x. Let

$$f(x) = \begin{cases} x\lfloor x \rfloor & \text{if } 1 \le x < 2 \\ 2x - 2 & \text{if } 2 \le x \le 3 \end{cases}.$$

   Which of the following statements are true about $f(x)$?

   I. $\lim_{x \to 2} f(x)$ exists.

   II. $f(x)$ is continuous at $x = 2$.

   III. $f(x)$ is differentiable at $x = 2$.

   IV. $\int_1^3 f(x)\,dx = 4$.

   (A) Only I (B) I and II (C) I, II, and III

   (D) I, II, and IV (E) none of these

5. Find a formula for $(f^{-1})'(x)$ given that f is one-to-one and $f'(x) = f(x)$.

   (A) $f(x) = \frac{1}{x}$ (B) $f(x) = e^x$ (C) $f(x) = x$ (D) $f(x) = 2^x$ (E) none of these

6. Consider the curves $y = x^n$, $n \in \mathbb{N}$, and suppose the tangent lines that go through $(1, 1)$ intersect with the x-axis at $(a_n, 0)$. Find $\lim_{n \to \infty} a_n^n$.

   (A) 0 (B) 1 (C) e (D) $\frac{1}{e}$ (E) none of these

7. Suppose the function $f(x) = ax^3 + bx^2 + cx + d$ has maximum value 1 at $x = 0$ and minimum value 0 at $x = 2$. Find a, b, c, d.

(A) $a = \frac{1}{2},\ b = -\frac{3}{2},\ c = \frac{1}{2},\ d = 1$ (B) $a = -\frac{1}{2},\ b = 1,\ c = 0,\ d = 1$

(C) $a = \frac{1}{4},\ b = -\frac{3}{4},\ c = 0,\ d = 1$ (D) $a = -\frac{1}{4},\ b = \frac{1}{2},\ c = 0,\ d = 1$ (E) none of these

8. Let $f(x)$ be continuous on $(-\infty, \infty)$ and let $g(x) = \int_{x^2}^{e^x} f(t)\, dt$. Find $g'(0)$.

(A) $f(0)$ (B) $f(1)$ (C) $-f(0)$ (D) $-f(1)$ (E) none of these

9. Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent line is parallel to the x-axis.

(A) $(2, 0), (-1, 27)$ (B) $(2, 0), (-\frac{5}{2}, 0)$ (C) $(0, 20), (-\frac{5}{2}, 0)$ (D) $(-2, 16), (1, 7)$ (E) none of these

10. Find the equation for the horizontal asymptote of $y = \dfrac{2\cos(x^2)}{x^2}$.

(A) $y = 2$ (B) $y = \frac{1}{2}$ (C) $y = 1$ (D) $y = 0$ (E) none of these

11. How many real roots does $f(x) = 4x^5 + x^3 + 7x - 2$ have?

(A) 0 (B) 1 (C) 2 (D) 3 (E) none of these

12. If $x \sin \pi x = \displaystyle\int_0^{x^2} f(t)\, dt$, where f is a continuous function, find $f(4)$.

(A) $f(4) = \pi$ (B) $f(4) = 2$ (C) $f(4) = \frac{\pi}{2}$ (D) $f(4) = \frac{\pi}{6}$ (E) none of these

13. Consider a third-degree polynomial $f(x) = ax^3 + bx^2 + cx + d$. What condition on the coefficients of f ensures the existence of two (2) distinct critical points?

(A) $b^2 > 3ac$ (B) $b^2 = 3ac$ (C) $b^2 < 3ac$ (D) $b^2 \geqslant 3ac$ (E) none of these

14. Suppose $f(x)$ is an odd function where $\lim\limits_{x \to 3^-} f(x) = 7$ and $\lim\limits_{x \to 3^+} f(x) = 5$. Then

(A) $\lim\limits_{x \to -3^-} f(x) = 5$ (B) $\lim\limits_{x \to -3^+} f(x) = 5$ (C) $\lim\limits_{x \to -3^-} f(x) = -5$

(D) $\lim\limits_{x \to -3^+} f(x) = -5$ (E) none of these

15. Find the point on the parabola $y^2 = 16x$ such that the distance from this point to the line $4x - 3y + 24 = 0$ is minimized.

(A) $\left(\frac{1}{4}, 2\right)$ (B) $(1, 4)$ (C) $(4, 8)$ (D) $(16, 16)$ (E) none of these

16. John owns a small food stand in the busy downtown of Chicago. He finds out that if he can sell a lunchbox for p dollars, then he can sell $(70 - 2p)$ lunch boxes. It costs John $(710 - 9x)$ dollars to produce x lunch boxes. How much does John need to charge for one lunch box in order to maximize his profit?

(A) 9 (B) 11 (C) 13 (D) 15 (E) none of these

17. A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 miles north of the intersection and the car is 0.8 miles to the east, the police determine with radar that the distance between them and the car is increasing at 20 miles per hour (mph). If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the speeding car?

(A) 65 mph (B) 70 mph (C) 75 mph (D) 80 mph (E) none of these

18. Let L be the tangent line to the graph of $y = f(x) = 4x^3 + x - 2$ at $x = 1$. Find the area enclosed by the graph of $y = f(x)$ and L.

(A) 18 (B) 27 (C) 36 (D) 45 (E) none of these

19. Consider the curve $\Gamma : x^2 + xy + y^2 = 12$. Which of the following statements are true about Γ?

I. Γ has a local maximum at the point $(2, -4)$.
II. Γ has vertical tangent lines at $(4, -2)$ and $(-4, 2)$
III. Γ has no inflection points.
IV. Γ is an oblique parabola.

(A) I and II (B) III and IV (C) II and III (D) I, III, and IV (E) none of these

20. The floor function $\lfloor x \rfloor$ gives the greatest integer that is smaller than or equal to x. Evaluate $\lim_{x \to 0^+} x^2 \lfloor \frac{1}{x} - \frac{1}{x^2} \rfloor$.

(A) The limit does not exist. (B) 1 (C) 0 (D) -1 (E) none of these

21. Find $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{\sqrt{1^2 + 2^3 + \cdots + i^3}}$.

(A) 1 (B) 2 (C) 3 (D) 4 (E) none of these

22. Let $f(x) = x^2 \ln x$. Find $f^{(2019)}(1)$.

(A) $-2 \cdot 2016!$ (B) $2 \cdot 2016!$ (C) $-2 \cdot 2017!$ (D) $2 \cdot 2017!$ (E) none of these

23. Compute the integral $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\sin x}{1+\cos x} + |x| \right) dx$.

(A) $\dfrac{\pi^2}{4}$ (B) 0 (C) π (D) 2 (E) none of these

24. Which of the following statements are true about the curve defined by $x = \dfrac{1}{t-1}$ and $y = \dfrac{1}{t^2-1}$?

I. The curve has a horizontal asymptote.
II. The curve has a vertical asymptote.
III. The curve has an oblique asymptote.

(A) Only I (B) Only II (C) I and II (D) II and III (E) none of these

25. Suppose $f(x)$ is a continuous function where $f\left(\dfrac{1}{2}\right) = 2$, $f(4) = -1$, and $\displaystyle\int_{\frac{1}{2}}^{4} f(t)\,dt = 3$. For all nonzero real numbers x, define $g(x) = \displaystyle\int_1^{x^3} f\left(\frac{t}{x}\right) dt$. Evaluate $g'(2)$.

(A) 4 (B) 2 (C) -2 (D) -4 (E) none of these

26. Evaluate $\displaystyle\int_0^{\ln 2} \frac{1}{1+e^x}\,dx$.

(A) $\ln\left(\dfrac{3}{2}\right)$ (B) $\ln\left(\dfrac{4}{3}\right)$ (C) $\ln\left(\dfrac{5}{4}\right)$ (D) $\ln\left(\dfrac{6}{5}\right)$ (E) none of these

27. Let $f(x)$ be an invertible and differentiable function where $x^2 - [f(x)]^3 = xf(x)$ for $x \geq 0$. Evaluate $\left.\dfrac{d}{dy}\left[f^{-1}(y)\right]\right|_{y=2}$.

(A) $\dfrac{3}{2}$ (B) $\dfrac{8}{3}$ (C) $\dfrac{15}{4}$ (D) $\dfrac{24}{5}$ (E) none of these

28. Let D be the region in the first quadrant enclosed by the curves $y = \sqrt{1-x^2}$ and $x^{\frac{3}{2}} + y^{\frac{3}{2}} = 1$. Find the volume of the solid that is obtained by rotating D about the x-axis.

(A) $\dfrac{18\pi}{35}$ (B) $\dfrac{\pi}{2}$ (C) $\dfrac{9\pi}{35}$ (D) $\dfrac{\pi}{4}$ (E) none of these

29. For an everywhere differentiable function $g(x)$, suppose $g(1) = 7, g'(1) = -3$, and $g''(1) > 0.5$. Which of the following must be true about $g(4)$?

(A) $g(4) = -2$ (B) $g(4) > -2$ (C) $g(4) < -2$ (D) $g(4) = 7$ (E) none of these

30. Evaluate $\int_1^e e^{x^2}\left(2x\ln x + \frac{1}{x}\right)dx$.

(A) e^{e^2} (B) 0 (C) $e^2 - 1$ (D) $2e - 1$ (E) none of these

31. Evaluate $\int_0^1 \left(\sum_{n=0}^{\infty}(-1)^n x^{2n}\right)dx$.

(A) The integral diverges. (B) 1 (C) $\pi/2$ (D) $\pi/4$ (E) none of these

32. Let S be the region in the first and the second quadrants in the xy-plane ($y \geq 0$) consisting of all the points point (x, y) that are closer to the origin than to the horizontal line $y = 1$. Find the area of S.

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) 1 (D) π (E) none of these

33. Let $f : [0, 1] \to [0, 1]$ be defined as follows:

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2 \\ 2 - 2x & \text{if } 1/2 < x \leq 1. \end{cases}$$

Suppose that a sequence of functions $f_n : [0, 1] \to [0, 1]$, with $n = 1, 2, 3, \ldots$, is defined by composing the function $f(x)$ with itself for $n - 1$ times, i.e., $f_1(x) = f(x)$, and for $n \geq 2$, $f_n(x) = f(f_{n-1}(x))$. (For example, $f_3(x) = f(f(f(x)))$, $f_4(x) = f(f(f(f(x))))$, and so on.) Find the following limit:

$$\lim_{n\to\infty}\left(\int_0^1 f_n(x)dx\right).$$

(A) ∞ (B) 1 (C) 1/2 (D) The limit does not exist. (E) None of these

34. Evaluate $\lim_{x\to\infty}\left(\sin\sqrt{x+\pi} - \sin\sqrt{x}\right)$.

(A) 0 (B) -1 (C) 1 (D) The limit does not exist. (E) none of these

35. Find $\lim_{n\to\infty} \cos\frac{1}{2}\cos\frac{1}{4}\cos\frac{1}{8}\cdots\cos\frac{1}{2^n}$.

(A) $\sin 1$ (B) $\cos 1$ (C) $\tan 1$ (D) $\cot 1$ (E) none of these

36. Compute the integral $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$

(A) 0 (B) $\frac{1}{2}$ (C) $\frac{\pi}{2}$ (D) $\frac{\pi^2}{4}$ (E) none of these

37. Find $\lim_{n\to\infty}(2^{\frac{1}{n}}-1)\sum_{i=0}^{n-1}2^{\frac{i}{n}}\sin(2^{\frac{2i+1}{2n}})$

(A) $\cos 1-\cos 2$ (B) $\cos 2-\cos 1$ (C) $\sin 1-\sin 2$ (D) $\sin 2-\sin 1$ (E) none of these

38. Evaluate $\int_0^{2\pi}\frac{dx}{2+\cos x}$.

(A) 0 (B) ∞ (C) $\frac{2\pi}{\sqrt{3}}$ (D) $\frac{\pi}{\sqrt{3}}$ (E) none of these

39. Consider the region S bounded between the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ and the axes in the first quadrant. Let m be a real number such that the straight line $y=mx$ cuts the region S into two halves with equal areas. Which of the following is the point of intersection of the ellipse and the line $y=mx$?

(A) $(\frac{5}{\sqrt{3}},2\sqrt{2})$ (B) $(3,\frac{12}{5})$ (C) $\left(\frac{\sqrt{3}}{2},\frac{\sqrt{5}}{2}\right)$ (D) $\left(\frac{5}{\sqrt{2}},\frac{3}{\sqrt{2}}\right)$ (E) none of these

40. Evaluate $\lim_{n\to\infty}\left(\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)\right)^{1/n}$.

(A) The limit does not exist (B) 0 (C) 1 (D) 2 (E) none of these